# Semiclassical description of confinement via center vortices and anomaly-preserving $T^2$ compactifications

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based on 2201.06166[hep-th]

# Introduction

Goal of this talk is to DERIVE

- Confinement of 4d SU(N) Yang-Mills theory, and
- Chiral effective Lagrangian for QCD,

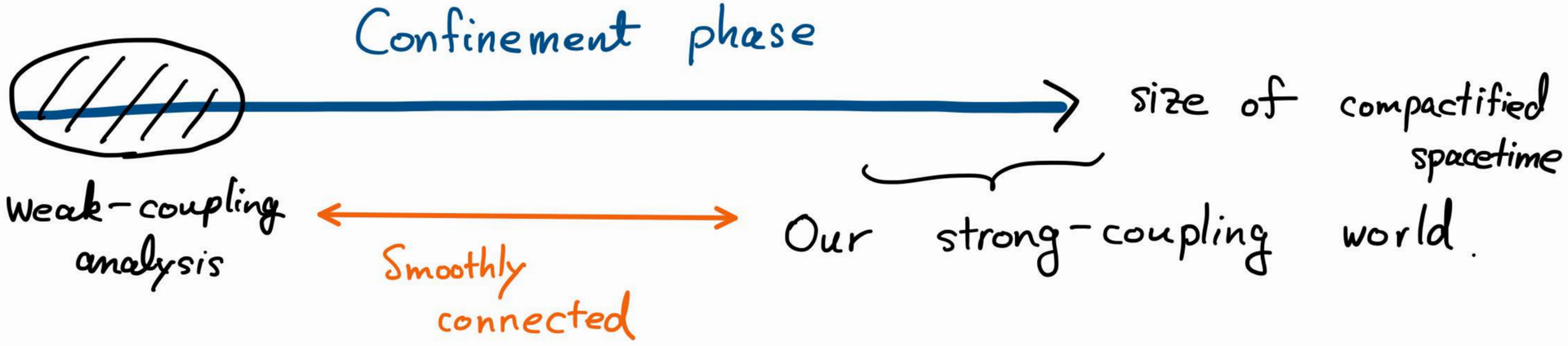
in a CALCULABLE setup with SEMICLASSICAL methods.

However, confinement seems to be a strong-coupling physics.

**Why can we expect that it's possible?**

# Adiabatic Continuity to Weak-Couplings.

We'd like to achieve:



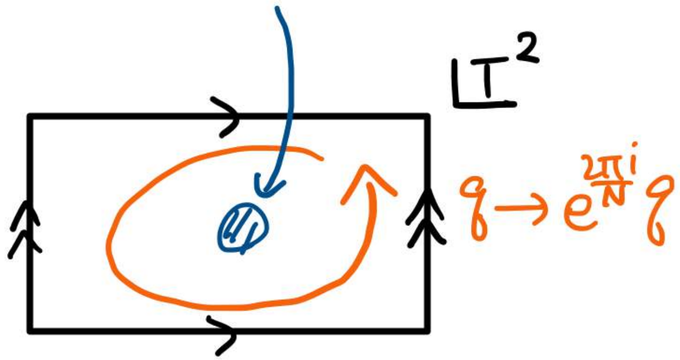

cf. · Twisted Eguchi-Kawai model (González-Arroyo, Okawa)

· Double-trace deformed YM
or YM theory w/ adjoint fermions
} on $\mathbb{R}^3 \times S^1$ (Ünsal, ... 2007–)

## Conjecture (YT, Ünsal, '22)

YM, QCD on $\mathbb{R}^2 \times T^2$

w/ 't Hooft flux




Weak-coupling description

$\Longleftrightarrow$

Adiabatic Continuity

YM, QCD on $\mathbb{R}^4$

Strong-couplings

via Center Vortices

# Justifications

For small $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux,

we can use the Dilute Gas Approximation w/ center vortices.

It predicts

- (YM theory) $E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- ($\mathcal{N}=1$ SYM) $\langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$
- (QCD w/ non-commuting flavor twist $(N_c = N_f = N)$) $\langle \mathrm{tr}_{cf}(\bar{\psi})\, \mathrm{tr}_{cf}(\psi) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$

(Discrete chiral SSB)

- (QCD w/ $U(1)_B$ monopole flux) $S_{eff} \sim \int \left( |d U|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \chi_{top} (i \ln \det U - \theta)^2 \right)$

$\eta'$ mass from Witten-Veneziano formula

# Plan

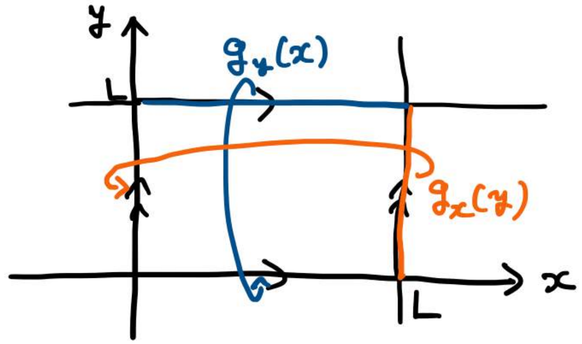
- Introduction
  - Confinement & adiabatic continuity
  - Our conjecture

- $T^2$ compactification with 't Hooft flux
  - Center vortex & confinement
  - Multi-branch structure of confinement vacua

YM theory

- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

't Hooft twisted boundary condition on $T^2 \ni (x, y)$

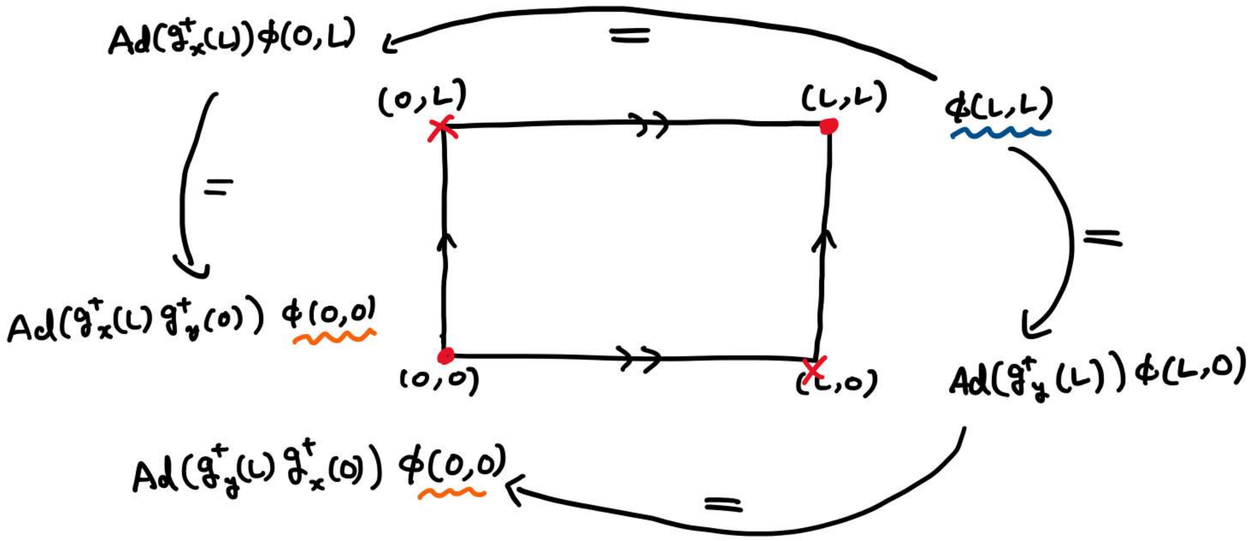



$$\begin{pmatrix} x \sim x+L \\ y \sim y+L \end{pmatrix}$$

$\phi(x, y)$ : adjoint matter field

$$\phi(L, y) = Ad\left(g_x^\dagger(y)\right) \phi(0, y)$$

$$\phi(x, L) = Ad\left(g_y^\dagger(x)\right) \phi(x, 0)$$




Uniqueness of the matter wavefunction requires

$$g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{\frac{2\pi i}{N} n_{xy}}$$

↑ 't Hooft flux.

( cf When fundamental matters exist, the condition becomes

$$g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0) \quad . \quad )$$

# Classical configuration and unbroken center symmetry

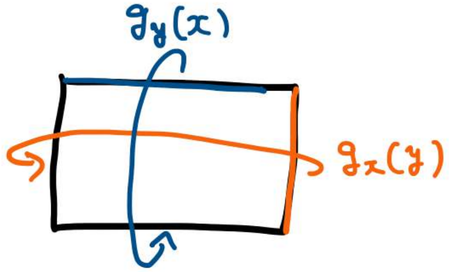
$F = 0$ gives the classical vacuum on $T^2$.

$\Rightarrow$ We can take a gauge so that $\boxed{a_\mu = 0}$.

However, Polyakov loops are forced to be nontrivial for the 't Hooft twist : $(n_{xy} = 1)$

$$g_x(y) \equiv S \propto \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

$$g_y(x) \equiv C \propto \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \quad (\omega = e^{\frac{2\pi i}{N}})$$




$(S, C \in SU(N)$ and $SC = CS\, e^{\frac{2\pi i}{N}})$.

$\Rightarrow P_x = S, \quad P_y = C.$

This is the **center-symmetric** configuration. $(\mathrm{tr}(P_x) = 0$ etc.$)$

Perturbative analysis of SU(N) YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

- $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken.

- 2d gluons are gapped.

  $\Leftarrow$ Polyakov loops along $T^2$ are adjoint Higgs fields for $\mathbb{R}^2$.

  $P_3 = S$, $P_4 = C$ gives

  $$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N.$$

  Weak-coupling analysis is free from IR divergences.

- However, Wilson loops inside $\mathbb{R}^2$ obey perimeter laws.

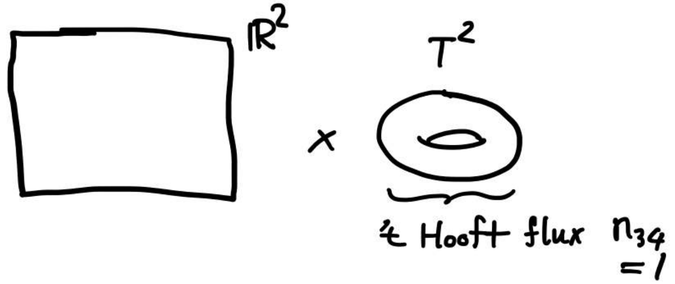
  $\Downarrow$

  We have to resolve this problem to achieve adiabatic continuity. $\Rightarrow$ Center vortex

# Center vortex as a fractional instanton on $\mathbb{R}^2 \times T^2$

In this setup, the minimal topological charge is given by

$$Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F \wedge F) = \frac{1}{N}$$



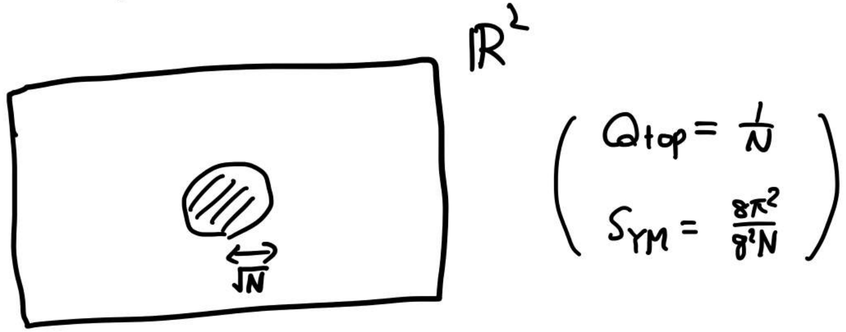

(More precisely, $Q_{top} \in \frac{1}{N}\left(\frac{-\epsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$S_{YM} = \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{g^2 \cdot N}$$

Gonzalez-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:

center vortex
or fractional instanton.




$$\left(\begin{array}{l} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{array}\right)$$

(cf. Garcia Perez, Gonzalez-Arroyo, '92, Itou '18)

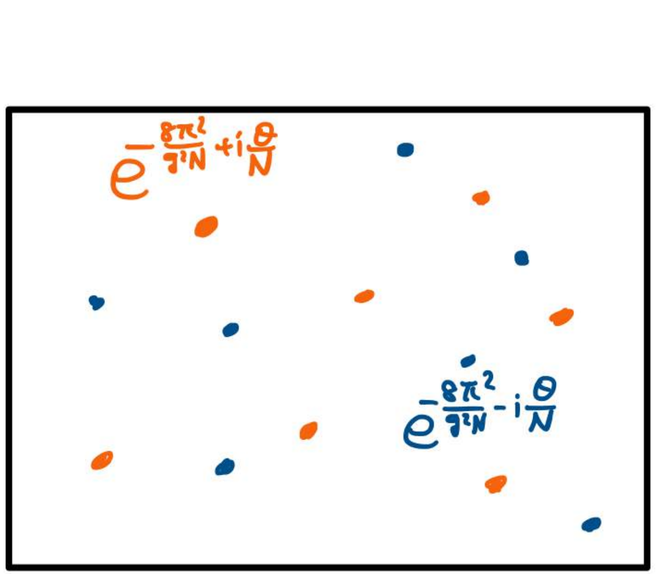
# Dilute gas approximation

2d gluon fields are perturbatively gapped by 't Hooft twist.

$\Rightarrow$ Center vortex, or fractional instanton, does NOT have the size moduli.

$\Rightarrow$ Dilute gas approximation is available.

(* In 4d pure YM, DIGA is invalidated because of IR divergences.)




$n$ : # of vortices

$\bar{n}$ : # of anti-vortices

$$Q_{top} = \frac{n-\bar{n}}{N}.$$

# Partition funcion on $M_2 \times T^2$ & $\theta$-dependence

$M_2 \to \mathbb{R}^2$

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold $M_2$.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \left(V \cdot K e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}\right)^n \left(V \cdot K e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}}\right)^{\bar{n}}$$

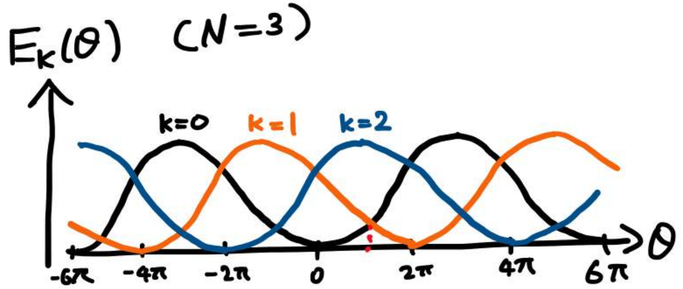
(vortex: $e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}$; anti-vortex: $e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}}$)

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta - 2\pi k}{N}\right)\right)\right]$$

$E_k(\theta)$ : Ground-state energy densities

$\Rightarrow$

- N-branch structure of ground states.
- Each branch has a fractional $\theta$-dependence.



# Anomaly matching in 4d and 2d

4d SU(N) YM theory has an 't Hooft anomaly : (Gaiotto, Kapustin, Komargodski, Seiberg '17)

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B}\, Z_\theta[B]$$

$\Longrightarrow$ Confinement implies the multi-branch structure of vacua.

With $T^2$-compactification, $(\mathbb{Z}_N^{[1]})_{4d}$ splits into $(\mathbb{Z}_N^{[1]})_{2d} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

When 't Hooft flux $n_{34}$ (mod N) is introduced, the 4d anomaly becomes

$$Z_{\theta+2\pi}[B_{2d}, A, A'] = e^{i\left(n_{34}\int B - \frac{N}{2\pi}\int A\wedge A'\right)}\, Z_\theta[B, A, A'].$$

$\Longrightarrow$ When $\gcd(n_{34}, N)=1$ (especially when $n_{34}=1$),

confinement implies the multi-branch structure. (cf. YT, Misumi, Sakai, '17. Yamazaki '17.)

Both properties are obtained by the center vortex.

# Hamiltonian picture

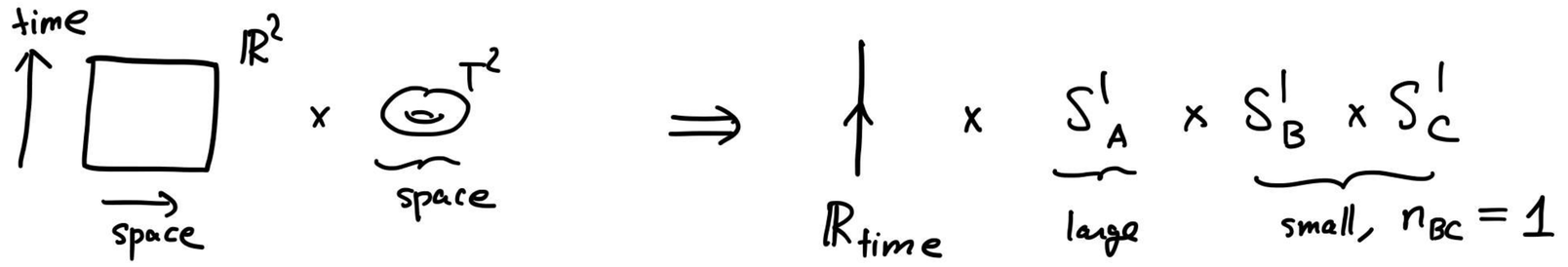
Using the dilute gas approximation of the YM path integral, we obtain

- N-branch structure of vacua
- Fractional $\theta$-dependence $E_k(\theta) \sim \Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$
- Confinement of Wilson loops for non-trivial N-alities.

To get a better understandings,

let us construct the wave-function of these confining vacua.

Setup




(cf. Yamazaki, Yonekura '17 : $L_A, L_B \ll L_C$ ($\ll \Lambda^{-1}$) w/ $n_{BC} = 1$)

Perturbative vacua

We now have three holonomies (Polyakov loops along $S^1_A$, $S^1_B$, $S^1_C$): $P_A, P_B, P_C$.

't Hooft flux $n_{BC} = 1$ : Classical vacuum $F_{BC} = 0$ gives

$$P_B = S \,, \qquad P_C = C \,.$$

We note that $n_{AB} = n_{AC} = 0$ : Classical vacuum $F_{AB} = F_{AC} = 0$ gives

$$P_A P_B = P_B P_A \,, \qquad P_A P_C = P_C P_A \,.$$

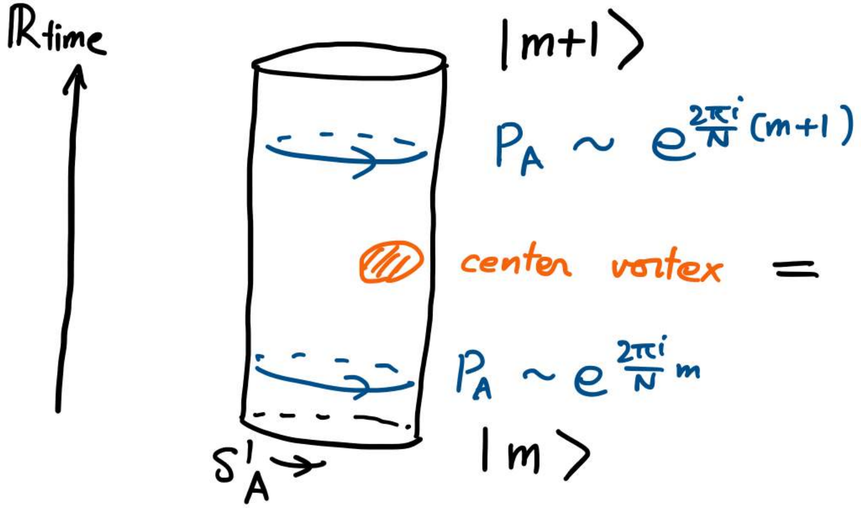
$$\Rightarrow \qquad P_A = e^{\frac{2\pi i}{N} m} \mathbb{1}_N \qquad (m = 0, 1, \cdots, N-1) \,.$$

Classically, $\mathbb{Z}_N$ center symmetry for $P_A$ is spontaneously broken,

and we get $N$ degenerate vacua.

$$|m\rangle \; : \; \frac{1}{N} \mathrm{tr}(P_A) \, |m\rangle = e^{\frac{2\pi i}{N} m} \, |m\rangle \,.$$

# Tunneling amplitude via center vortex




= Tunneling from $|m\rangle$ to $|m+1\rangle$.

Within the one-center-vortex approximation,

$$\langle n | \exp(-T\hat{H}_{YM}) | m \rangle$$

$$\sim \underbrace{\delta_{nm}}_{\text{No tunneling}} + T \cdot L_A \cdot K e^{-\frac{8\pi^2}{g^2 N}} \left( \underbrace{e^{i\frac{\theta}{N}} \delta_{n,m+1}}_{\substack{\text{tunneling via} \\ \text{center vortex}}} + \underbrace{e^{-i\frac{\theta}{N}} \delta_{n,m-1}}_{\substack{\text{tunneling via} \\ \text{anti center vortex}}} \right)$$

Especially, $|m\rangle$ is not an eigenstate of $\hat{H}_{YM}$.

## Eigenstate of $\exp(-T\hat{H}_{YM})$ and confining vacua

Let us consider the following wavefunction

$$|\widetilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} km} |m\rangle .$$

This is an eigenstate of $\exp(-T\hat{H}_{YM})$:

$$\exp(-T\hat{H}_{YM})\,|\widetilde{k}\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{n,m} e^{\frac{2\pi i}{N} km} |n\rangle \underbrace{\langle n| e^{-T\hat{H}_{YM}} |m\rangle}_{\sim\ \delta_{nm} + TL_A\, k e^{-\frac{8\pi^2}{g^2N}} \left(e^{i\frac{\theta}{N}} \delta_{n,m+1} + e^{-i\frac{\theta}{N}} \delta_{n,m-1}\right)}$$

$$\sim \exp\left[-T\cdot L_A \cdot \underbrace{\left(-2k e^{-\frac{8\pi^2}{g^2N}} \cos\left(\frac{\theta-2\pi k}{N}\right)\right)}_{= E_k(\theta)}\right] \cdot |\widetilde{k}\rangle .$$

$\Rightarrow |\widetilde{k}\rangle$ give the wavefunction of confining vacua for $\mathbb{R}^2 \times \underbrace{T^2}_{\text{'t Hooft flux}}$.

# Plan

- Introduction
  - Confinement & adiabatic continuity
  - Our conjecture
- $T^2$ compactification with 't Hooft flux
  - Center vortex & confinement
  - Multi-branch structure of confinement vacua
- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

QCD

't Hooft flux with fundamental quarks

We have seen that

Semiclassical theory w/ center vortices

$\Rightarrow$ Confinement vacua of YM theory.

Next, we would like to apply this idea for

**Chiral effective Lagrangian of QCD.**

Obstacle: We cannot introduce the 't Hooft flux when fundamental matters exist.

$U(1)_B$ vs $U(1)_q$

$$\text{Baryon numbers} = \frac{\text{quark numbers}}{N}$$

i.e. $$U(1)_B = U(1)_q / \mathbb{Z}_N .$$

Let us denote $U(1)_B$, $U(1)_q$ – gauge fields as $A_B$, $A_q$ :

$$\frac{1}{N} A_B = A_q .$$

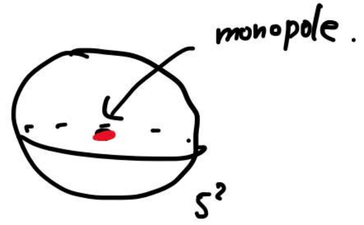
$$\left( \Leftarrow \quad A_{B\mu} J_B^\mu = A_{q\mu} J_q^\mu \quad \text{and} \quad J_B^\mu = \frac{1}{N} J_q^\mu \right)$$

However, this naive relation looks to be problematic when we consider monopoles :

$U(1)_B$ monopole : $\int_{S^2} dA_B = 2\pi .$ $\left( \Leftrightarrow \int_{S^2} dA_q = \frac{2\pi}{N} \right)$

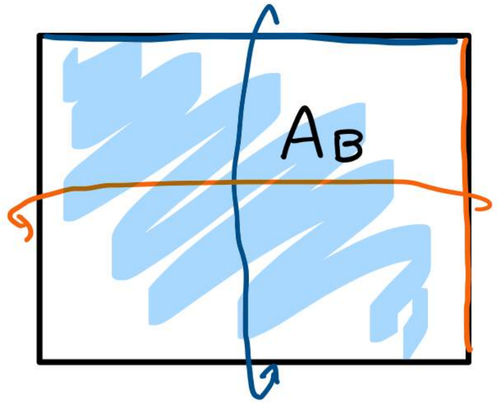
$U(1)_q$ monopole : $\int_{S^2} dA_q = 2\pi$

normalization doesn't match.




$U(1)_B$ monopole violates Dirac quantization.

# $U(1)_B$ monopole flux & 't Hooft flux on $T^2$




quark field

$$\begin{cases} \psi(L,y) = g_x^\dagger(y)\, e^{-i\frac{\phi_x(y)}{N}}\, \psi(0,y) \\ \psi(x,L) = g_y^\dagger(x)\, e^{-i\frac{\phi_y(x)}{N}}\, \psi(x,0) \end{cases}$$

color-transition functions

$U(1)_g$-transition functions

Cocycle condition

$$g_x^\dagger(L)\, g_y^\dagger(0)\, e^{-i\frac{1}{N}(\phi_x(L)+\phi_y(0))} = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{-i\frac{1}{N}(\phi_y(L)+\phi_x(0))}$$

$U(1)_B$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_x(L)-\phi_x(0)) - (\phi_y(L)-\phi_y(0))$$

$$\Longrightarrow \quad g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}} \qquad \text{(cf. YT, '18)}$$

't Hooft flux !!

# Perturbative spectrum

Gauge fields : Same with SU(N) YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

$\Rightarrow \mathbb{Z}_N$ gauge field.

Quark fields : Solve the Dirac zero-mode eq.

$$\left[\gamma_3\left(\partial_3 + i\frac{1}{N}A_{B,3}\right) + \gamma_4\left(\partial_4 + i\frac{1}{N}A_{B,4}\right)\right]\psi = 0.$$

$\Rightarrow$ For each 4d fundamental Dirac fermion,

there is a 2d massless Dirac fermion.

For $N_f$ - flavor massless QCD, we have 2d $N_f$ Dirac fermions.

$\longleftrightarrow$ non-Abelian bosonization $U(N_f)_1$ WZW model $\frac{1}{8\pi}\int_{M_2} \mathrm{tr}\,(dU^\dagger \wedge dU) + \frac{1}{12\pi}\int_{M_3} \mathrm{tr}\,[(U^\dagger dU)^3]$

w/ $U(N_f)$-valued field $U : M_2 \to U(N_f)$.

## Center-vortex induced mass for $\eta'$

Let us construct the center-vortex vertex $\sim e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}$.

$U(1)$ axial anomaly requires the spurious symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}, \quad \theta \to \theta + 2N_f \alpha.$$

In the bosonized description

$$U \to e^{-2i\alpha} U.$$

$$\Rightarrow \quad e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}} \cdot (\det U)^{\frac{1}{N}} \quad \text{satisfies}$$

- $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry,
- $U(1)$ axial anomaly relation.

$$\Rightarrow \Delta S_{\text{center-vortex}} \sim \underbrace{\Lambda^2 (\Lambda L)^{\frac{5N-2N_f}{3N}}}_{\eta' \text{ mass}} \cdot \cos\left(\frac{i \ln(\det U) - \theta}{N}\right)$$

# SUMMARY

YM, QCD on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux $\Longleftrightarrow$ YM, QCD on $\mathbb{R}^4$

Adiabatic Continuity

Semiclassics w/ center vortices

Strong-couplings

- (YM theory) $E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- ($\mathcal{N}=1$ SYM) $\langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$
- (QCD w/ non-commuting flavor twist ($N_c = N_f = N$)) $\langle \mathrm{tr}_{cf}(\bar{\psi})\, \mathrm{tr}_{cf}(\psi) \rangle \sim \Lambda^3 e^{i\frac{\theta - 2\pi k}{N}}$

(Discrete chiral SSB)

- (QCD w/ $U(1)_B$ monopole flux) $S_{eff} \sim \int \left( |d U|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \chi_{top} (i \ln \det U - \theta)^2 \right)$

$\eta'$ mass from Witten-Veneziano formula